Calculus – Lesson 3: Rational Functions (page 1)
Upward Bound Summer 2018 **Name: ____________________**

Basic: Simplifying and Combining Rational Expressions

A **rational expression** is a fraction with a polynomial numerator and nonzero polynomial denominator. We can simplify rational expressions by factoring the polynomial on the top and the bottom and cancelling any common factors.

For example, $\frac{x^2+3x+2}{x^2-1} = \frac{(x+1)(x+2)}{(x+1)(x-1)} = \frac{x+2}{x-1}$. If we want to be super-technical, the simplified expression should still have the same domain as the original expression, so we might write the simplification as $\frac{x+2}{x-1},\ x \neq -1$.

Simplify each rational expression:

Prob. #	Original Expression	Simplified Expression
1	$\frac{x^2-6x+8}{x^2-16}$	
2	$\frac{3(x+4)(x+1)x}{2(x+3)(x+1)^2x}$	
3	$\frac{3x^3+6x^2}{x^2+4x+4}$	
4	$\frac{(2x)^4}{2x^4}$	
5	$\frac{x^4-1}{x^2+1}$	
6	$\frac{(x+2)x+4(x-1)}{x^2-9}$	

You can add and subtract rational expressions just as you would add and subtract fractions – get a common denominator and then add or subtract the numerators.

For example, $\frac{x}{x+1} - \frac{x+5}{x-2} = \frac{x(x-2)-(x+5)(x+1)}{(x+1)(x-2)} = \frac{(x^2-2x)-(x^2+6x+5)}{(x-1)(x-2)} = \frac{-8x-5}{(x-1)(x-2)}$.

You can also multiply and divide rational expressions as you would multiply and divide fractions.

For example, $\frac{x}{x+1} \div \frac{x+3}{x^2-1} = \frac{x}{x+1} \cdot \frac{x^2-1}{x+3} = \frac{x(x+1)(x-1)}{(x+1)(x+3)} = \frac{x(x-1)}{x+3}$.

For each pair of rational expressions on the next page, find the sum, difference, product, and quotient. Remember to simplify your answers.

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Prob. #	RE1	RE2	RE1+RE2	RE1-RE2	RE1*RE2	RE1/RE2
7	$\frac{1}{x}$	$\frac{1}{x-1}$				
8	$\frac{2}{x+4}$	$\frac{2}{x-4}$				
9	$\frac{x^2-2x+1}{x+5}$	$\frac{x^2+4x-5}{x+1}$				
10	$\frac{1}{x}$	$\frac{1}{x^2}$				

Intermediate: Graphing Rational Functions

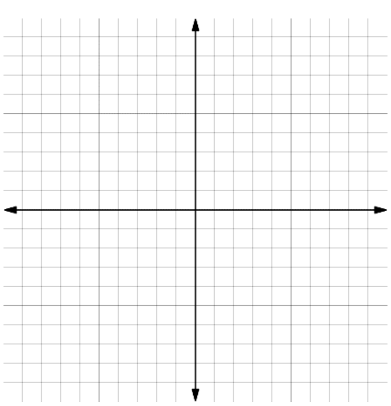
Vertical Asymptotes: Vertical lines which the graph of a function approaches but does not touch. These can be found by setting the denominator of the *simplified* fraction equal to zero and solving for x. Remember that these are *lines*, so you should write your vertical asymptotes in the form $x = h$, not simply h.

Holes: Points which are on the graph of the simplified fraction but technically should not be on the graph because of a $\frac{0}{0}$ error. To find the x-coordinates of holes, find all values of x for which the *original* denominator is equal to zero; the ones which are not vertical asymptotes are holes. You can then find the corresponding y-coordinate for each hole by plugging the x-coordinate into the simplified fraction.

Horizontal and Slant Asymptotes: Horizontal and diagonal lines which the graph of a function approaches but does not touch. These are the quotient you obtain by performing long division on the fraction. In the case of horizontal asymptotes, this means you can just divide the leading coefficients. Remember that these are *lines*, so you should write your horizontal asymptotes in the form $y = k$, not simply k.

Graph each rational function below to the best of your current ability.

(11) $f(x) = \frac{1}{x+3}$



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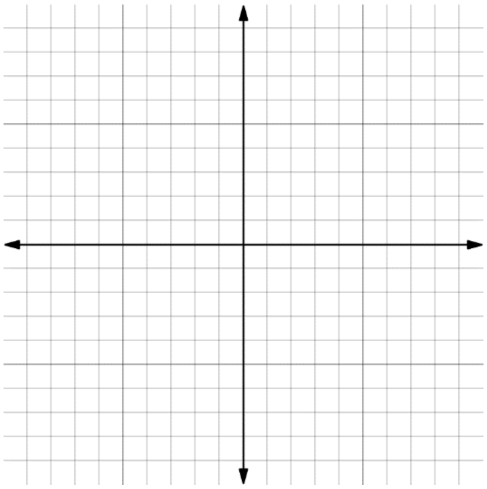
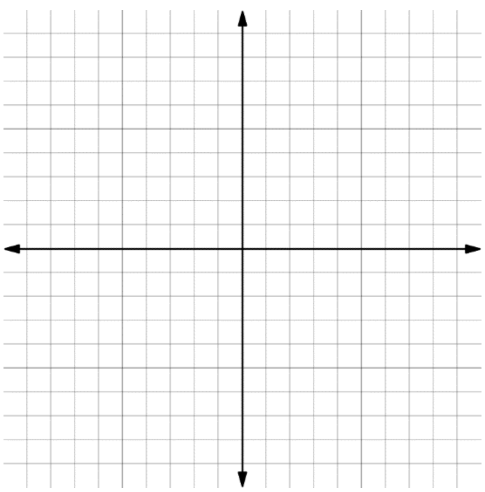
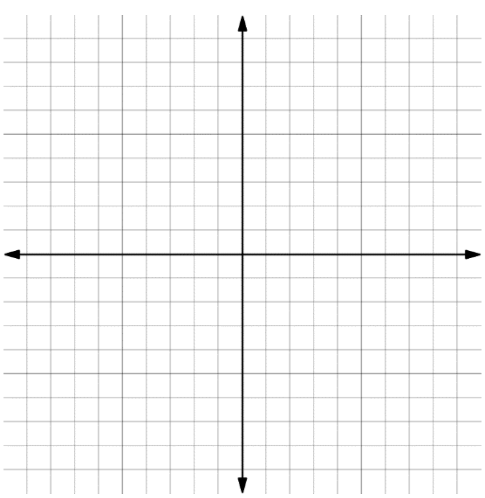
(12) $g(x) = 10 - \frac{2}{x-4}$



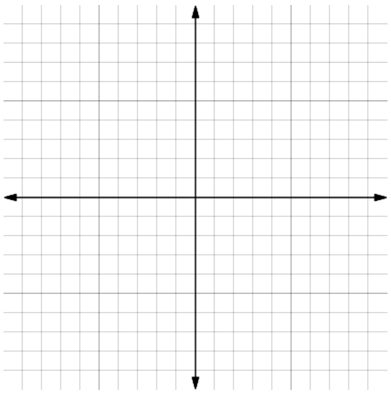
(13) $h(x) = \frac{x^2-6x+8}{x^2-16}$



(14) $f(x) = \frac{3(x+4)(x+1)x}{2(x+3)(x+1)^2x}$



(15) $g(x) = \frac{x^2+3x-5}{x-1}$



Given information about a rational function's graph, can you determine its equation?

(16) Write a rational function which has a hole at $x = 2$, a vertical asymptote $x = -2$, a horizontal asymptote $y = 4$, and an x-intercept at $x = 5$.

<u>Advanced: What can partial fractions tell us about the whole graph?</u>

Partial Fractions is a method of understanding a rational expression by rewriting it as a sum of simpler rational expressions. You will use this much later in Calculus for another purpose, but for now let's look at what Partial Fractions can show us about the graphs of rational functions.

Let f be the rational function defined by $f(x) = \frac{-x+7}{x^2+6x+5}$.

(17) Find numbers A and B such that if $g(x) = \frac{A}{x+1}$ and $h(x) = \frac{B}{x+5}$ then $f = g + h$.

(18) Graph f, g, and h on the same grid. What comparisons can you make between the characteristics of the graph of f and those of the graphs of g and h?